ELEMENTARY CONCEPTS AND OBJECTIVES OF OPERATIONS RESEARCH

Introduction

Operations Research (OR) is relatively a new discipline. OR provides a quantitative technique or a scientific approach to the executives for making better decisions for operations under their control.

Definitions of Operations Research

Literally the word operation is defined as some action that we apply to some problems and the word Research means searching for new insight into the problem.

1. OR is a scientific method of providing executive departments with a quantitative basis for decisions under their control.

   - P.M. Morse and G.E. Kimball

2. OR is the application of scientific methods, techniques and tools to problems involving the operations of a system so as to provide those in control of the system with optimum solutions to the problems.

   - Churchman, Ackoff and Arnoff

3. OR is applied decision theory. It uses any scientific, mathematical or logical means to attempt to cope with the problems that confront the executive, when he tries to achieve a thorough going rationality in dealing with his decision problems.

   - D.W. Miller and M.K. Starr

4. OR is the attack of modern science on problems of likelihood that arise in the management and control of men and machines, materials and money in their natural environment, its special technique is to invent a strategy of control by measuring, comparing and predicting probable behavior through a scientific model of a situation.

   - Beerr

5. OR is a scientific method of providing the executive with an analytical and objective basis for decisions.

   - P.M.S. Blackett

6. The term OR connotes various attempts to study operations of war by scientific methods. From a more general point of view OR can be considered to be an attempt to study these operations of modern society which involve organizations of men or of men and machines.

   - P.M. Morse

7. OR is a management activity pursued in two complementary ways one half by the free and bold exercise of commonsense untrammeled by any routine and other half by the application of a repertoire of well established pre-created methods and techniques.

- Jagjit Singh

8. OR is the attack of modern science on complex problems arising in the direction and management of large systems of men, machines, materials and money in industry, business and defence. The distinctive approach is to develop a scientific model of the system, incorporating measurements of factors such as chance and risk with which to predict and compare the outcomes of alternative decisions, strategies or controls. The purpose is to help management to determine its policies and actions scientifically.

- Operational Research Quarterly

9. OR is the art of giving bad answers to problems which otherwise have worse answers.

- T.L. Saaty

From the above opinions, it may be generalized that OR is certainly concerned with optimization theory.

Scope of Operations Research

OR is mainly concerned with the techniques of applying scientific knowledge, besides the development of science. It provides an understanding which gives the expert/manager new insights and capabilities to determine better solutions in his decision making problems, with great speed, competence and confidence. OR has been found to be used in the following five major areas of research:

1. OR is useful to the Directing Authority in deciding optimum allocation of various limited resources such as men, machines, material, time, money, etc., for achieving the optimum goal.
2. OR is useful to Production Specialist in
   i. Designing, selecting and locating sites.
   ii. Determining the number and size.
   iii. Scheduling and sequencing the production runs by proper allocation of machines; and
   iv. Calculating the optimum product mix.
3. OR is useful to the Marketing Manager (executive) in determining:
   i. How to buy, how often to buy, when to buy and what to buy at the minimum possible cost.
   ii. Distribution points to sell the products and the choice of the customers.
   iii. Minimum per unit sale price.
   iv. The customers preference relating to the size, colour, packaging etc., for various products and the size of the stock to meet the future demand; and

v. The choice of different media of advertising.

4. OR is useful to the Personnel Administrator in finding out:
    i. Skilled persons at a minimum cost.
    ii. The number of persons to be maintained on full time basis in a variable work load like freight handling etc.; and
    iii. The optimum manner of sequencing personnel to a variety of jobs.
5. OR is useful to the Financial Controller to
    i. Find out a profit plan for the company.
    ii. Determine the optimum replacement policies.
    iii. Find out the long-range capital requirements as well as the ways and means to generate these requirements.

Introduction

A large number of business and economic situations are concerned with problems of planning and allocation of resources to various activities. In each case there are limited resources at our disposal and our problem is to make such a use of these resources so as to maximize production or to derive the maximum profit, or to minimize the cost of production . Such problems are referred as the problems of optimization. Linear programming (LPP) is one of the most popular and widely used quantitative techniques. Linear Programming is a technique for determining an optimum schedule chosen from a large number of possible decisions. The technique is applicable to problem characterized by the presence of a number of decision variables, each of which can assume values within a certain range and affect their decision variables. The variables represent some physical or economic quantities which are of interest to the decision maker and whose domain are governed by a number of practical limitations or constraints which may be due to availability of resources like men, machine, material or money or may be due quality constraint or may arise from a variety of other reasons. The most important feature of linear programming is presence of linearity in the problem. The word Linear stands for indicating that all relationships involved in a particular problem are linear. Programming is just another word for planning and refers to the process of determining a particular plan of action from amongst several alternatives. The problem thus reduces to maximizing or minimizing a linear function subject to a number of linear inequalities

Definitions of Various Terms Involved in Linear Programming

Linear

The word linear is used to describe the relationship among two or more variables which are directly proportional. For example, if the production of a product is proportionately increased, the profit also increases proportionately, then it is a linear relationship. A linear form is meant a mathematical expression of the type,

$a_1X_1 + a_2X_2 + -- +a_nX_n$ *where* $a_1, a_2, ---, a_n$ *are constant and* $X_1, X_2, ---, X_n$ *are variables.*

Programming

The term Programming refers to planning of activities in a manner that achieves some optimal result with resource restrictions. A programme is optimal if it maximizes or minimizes some measure or criterion of effectiveness, such as profit, cost or sales.

Decision variables and their relationship

The decision variables refer to products or services that are competing with one another for sharing the given limited resources. These variables are usually inter-related in terms of utilisation of resources and need simultaneous solutions. The relationship among these variables should be linear.

Objective function

The Linear Programming problem must have a well defined objective function for optimization. For example, maximization of profits or minimization of costs is being studied. It should be expressed as linear function of decision variables.

Constraints

There are always limitations on the resources which are to be allocated among various competing activities. These resources may be production capacity, manpower, time, space or machinery. These must be capable of being expressed as linear equalities or inequalities in terms of decision variables.

Non-negativity restriction

All the variables must assume non-negative values, that is, all variables must take on values equal to or greater than zero. Therefore, the problem should not result in negative values for the variables as negative values of decision variables has no physical interpretation

Linearity and divisibility

All relationships (objective functions and constraints) must exhibit linearity, that is, relationships among decision variables must be directly proportional. For example, if our resources increase by some percentage, then it should increase the outcome by the same percentage. Divisibility means that the variables are not limited to integers. It is assumed that decision variables are continuous, i.e., fractional values of these variables must be permissible in obtaining an optimal solution.

Deterministic

In LP model (objective functions and constraints), it is assumed that the entire model coefficients are completely known (deterministic), e.g. profit per unit of each product, and amount of resources available are assumed to be fixed during the planning period.

Formulation of a Linear Programming Problem

The formulation of the Linear Programming Problem (LPP) as mathematical model involves the following key steps:

Step 1. Identify the decision variables to be determined and express them in terms of algebraic symbols as $X_1, X_2, \text{---}, X_n$.

Step 2. Identify the objective which is to be optimized (maximized or minimized) and express it as a linear function of the above defined decision variables.

Step 3. Identify all the constraints in the given problem and then express them as linear equations or inequalities in terms of above defined decision variables.

Step 4. Non-negativity restrictions on decision variables.

The formulation of a linear programming problem can be illustrated through the following examples:

Example 1

A dairy plant manufactures two types of products A and B and sells them at a profit of Rs. 5 on type A and Rs. 3 on type B. Each product is processed on two machines G and H. Type A requires one minute of processing time on G and two minutes on H; type B requires one minute on G and one minute on H. The machine G is available for not more than 6 hours 40 minutes, while machine H is available for 8 hours 20 minutes during any working day; formulate the problem as LP problem.

Solution

Let X_1 be the number of products of type A and X_2 the number of products of type B. Since the profit on type A is Rs. 5 per product, $5X_1$ will be the profit on selling X_1units of type A. Similarly, $3X_2$ will be the profit on selling X_2 units of type B. Therefore, total profit on selling X_1 units of A and X_2 units of B is given by $Z = 5X_1+3X_2$. Since this has to be maximized, hence the objective function can be expressed as Max $Z = 5X_1+3X_2$ (Objective function)

Since machine G takes 1 minute time on type A and 1 minute time on type B, the total time in minutes required on machine G is given by X_1+X_2. Similarly the total time in minutes required on machine H is given by $2X_1+X_2$. But machine G is not available for more than 6 hour 40 minutes (400 minutes), therefore

$X_1+X_2 \leq 400$ (first constraint on machine G)

Also the machine H is available for 8 hours 20 minutes only, therefore

$2X_1+X_2 \leq 500$ (second constraint on machine H)

Since it is not possible to produce negative quantities,

$X_1 \geq 0$, $X_2 \geq 0$ (non-negativity restrictions)

Example 2

A Milk plant packs two types of milk in pouches viz., full cream and single toned. There are sufficient ingredients to make 20,000 pouches of full cream & 40,000 pouches of single toned. But there are only 45,000 pouches into which either of the products can be put. Further it takes three hours to prepare enough material to fill 1000 pouches of full cream milk & one hour for 1000 pouches of single toned milk and there are 66 hours available for this operation. Profit is Rs. 8 per pouch for full cream milk and Rs. 7 per pouch for single toned milk. Formulate it as a linear programming problem.

Solution

Let number of full cream and single toned milk pouches to be packed is X_1 and X_2

Let profit be Z

Objective function: Max. $Z = 8X_1+7X2$

Subject to constraints: $X_1 + X_2 \leq 45000$

$$X_1 \leq 20000$$

$$X_2 \leq 40000$$

$$3\frac{X_1}{1000} + \frac{X_2}{1000} \leq 66 \implies 3X_1 + X_2 \leq 66000$$

Non-negativity restrictions $X_1, X_2 \geq 0$

Example 3

Consider two different types of food stuffs say F_1 and F_2. Assume that these food stuffs contain vitamin A and B. Minimum daily requirements of vitamin A and B are 40mg and 50mg respectively. Suppose food stuff F_1 contains 2mg of vitamin A and 5mg of vitamin B while F_2 contains 4mg of vitamin A and 2mg of vitamin B. Cost per unit of F_1 is Rs. 3 and that of F_2 is Rs. 2.5. Formulate the minimum cost diet that would supply the body at least the minimum requirements of each vitamin.

Solution

Let number of units needed for food stuffs F_1 and F_2 to meet the daily requirements of vitamins A and B be respectively X_1 and X_2..

Objective function: Minimize $Z = 3X_1+2.5X_2$

Subject to constraints:

$$2X_1+4X_2 \geq 40$$

$$5X_1+2X_2 \geq 50$$

Non-negativity restrictions $X_1, X_2 \geq 0$

Mathematical Formulation of a General Linear Programming Problem

The general formulation of LP problem can be stated as follows. If we have n-decision variables X_1, X_2, X_n and m constraints in the problem , then we would have the following type of mathematical formulation of LP problem

Optimize (Maximize or Minimize) the objective function:

$$Z = C_1X_1 + C_2X_2 + \ + C_nX_n$$

Subject to satisfaction of m- constraints:

$$\left.\boxed{\begin{array}{l} a_{11} X_1 + a_{12} X_2 + \dots + a_{1n} X_n \ (\leq = \geq)\ b_1 \\ a_{21} X_1 + a_{22} X_2 + \dots + a_{2n} X_n \ (\leq = \geq)\ b_2 \\ a_{i1} X_1 + a_{i2} X_2 + \dots + a_{in} X_n \ (\leq = \geq)\ b_i \\ a_{m1} X_1 + a_{m2} X_2 + \dots + a_{mn} X_n \ (\leq = \geq)\ b_m \end{array}}\right\} \qquad \text{......... (Eq.3.4.2)}$$

Where the constraint may be in the form of an inequality ($\leq$ or $\geq$) or even in the form of an equality (=) and finally satisfy the non-negativity restrictions

$$X_1, X_2, \ , X_n \geq$$

where C_j (j=1, 2,.,n); b_i (i=1, 2,m) and a_{ij} are all constants and m<n, and the decision variables $X_j \geq 0$, j=1, 2,,n. If b_i is the available amount of resource i then a_{ij} is amount of resource i that must be allocated (technical coefficient) to each unit of activity j.

NOTE: By convention, the values of RHS parameters b_i (i=1, 2, 3,,m) are restricted to non-negative values only. If any value of b_i is negative then it is to be changed to a positive value by multiplying both sides of the constraint by -1. This not only changes the sign of all LHS Coefficients and of RHS parameters but also changes the direction of inequality sign.

Matrix form of LP problem

The linear programming problem can be expressed in matrix form as

Maximize $Z = CX$ (objective function)

Subject to $AX = b$, $b \geq 0$ (constraint equation)

$X \geq 0$ (non-negativity restriction)

Where $X = (X_1, X_2, \dots X_n)$, $C = (C_1, C_2, \dots, C_n)$ and $b = (b_1, b_2, \dots, b_m)$

$$A = \begin{pmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ a_{m1} & a_{m1} \cdots & a_{mn} \end{pmatrix}$$

Graphical Solution of Linear Programming Problem

LP problems which involve only two variables can be solved graphically. Such a solution by geometric method involving only two variables is important as it gives insight into more general case with any number of variables.

Feasible solution

A set value of the variables of a linear programming problem which satisfies the set of constraints and the non-negative restrictions is called feasible solution of the problem.

Feasible region

The collection of all feasible solutions is known as the feasible region. Any point which does not lie in the feasible region cannot be a feasible solution to the LP problems. The feasible region does not depend on the form of the objective function in any way. If we can represent the relations of the general LP problem on an n dimensional space, we will obtain a shaded solid figure (known as Convex-polyhedron) representing the domain of the feasible solution.

Optimal solution

A feasible solution of a linear programming problem which optimizes its objective function is called the optimal solution of the problem. Theoretically, it can be shown that objective function of a LP problem assumes its optimal value at one of the vertices (called extreme points) of this solid figure.

Steps to find graphical solution of the linear programming problem

- **Step 1:** Formulate the linear programming problem.
- **Step 2:** Draw the constraint equations on XY-plane.
- **Step 3:** Identify the feasible region which satisfies all the constraints simultaneously. For less than or equal to constraints the region is generally below the lines and for greater than or equal to constraints, the region is above the lines.
- **Step 4:** Locate the solution points on the feasible region. These points always occur at the vertices of the feasible region.
- **Step 5:** Evaluate the optimum value of the objective function.

The geometric interpretation and solution for the LP problem using graphical method is illustrated with the help of following examples

Example 4 Find the graphical solution of problem formulated in example 1.

Max $Z = 5X_1+3X_2$

Subject to

$X_1+X_2 \leq 400$

$2X_1+X_2 \leq 500$

$X_1 \geq 0$, $X_2 \geq 0$

Solution

Any point lying in the first quadrant has X_1, $X_2 \geq 0$ and hence satisfies the non-negativity restrictions. Therefore, any point which is a feasible solution must lie in the first quadrant. In order to find the set of points in the first quadrant which satisfy the constraints, we must interpret geometrically inequalities such as $X_1+X_2 \leq 400$. If equality holds then we have the equation $X_1+X_2= 400$ i.e., any point on the straight line satisfies the equation. Any point lying on or below the line $X_1+X_2= 400$ satisfies the constraint $X_1+X_2 \leq 400$. However, no point lying above the line satisfies the inequality. In a similar manner, we can find the set of points satisfying $2X_1+X_2 \leq 500$ and the non-negativity restrictions are all the points in the first quadrant. The set of points satisfying the constraints and the non-negativity restrictions is the set of points in the shaded region of the figure OABC as shown in Fig 3.1 Any point in this region is a feasible solution, and only the points in this region are feasible solutions. To solve the LP problem, we must find the point or points in the region of feasible solutions which give the largest value of the objective function. Now for any fixed value of Z, $Z = 5X_1+3X_2$ is a straight line, for each different values of Z, we obtain a different line. Again, all the lines corresponding to different values of Z are parallel; clearly our interest is to find the line with the largest value of Z which has at least one point in common with the region of feasible solutions. The line $Z= 5X_1+3X_2$ is drawn for various values of Z. It can be seen that the point farthest from $Z = 5X_1+3X_2$ and yet in the feasible region is B (100, 300) and thus point B maximizes Z while satisfying all the constraints. In other words, the corner B of the region of feasible solutions is the optimal solution of the LP problem. Since B is the intersection of the lines $X_1+X_2=400$ and $2X_1+X_2=500$, it can be seen that at vertex B, $X_1=100$ and $X_2=300$. The maximum value of $Z = 5(100)+3(300)=1400$.The fact that the optimum occurred at the vertex B of the feasible region is not a coincidence but on the other hand represents a significant property of optimal solutions of all LP problems. To find the optimal solution find the values of objective function at the various extreme points as shown in the following Table 3.1

Table 3.1 Computation of maximum value of objective function

Extreme Point	Coordinates	Profit Function $Z = 5X_1+3X_2$
O	$X_1=0$, $X_2=0$	$Z=5(0)+3(0)=0$
A	$X_1=0$, $X_2=400$	$Z=5(0)+3(400)=1200$
B	$X_1=100$, $X_2=300$	$Z=5(100)+3(300)=1400$
C	$X_1=250$, $X_2=0$	$Z=5(250)+3(0)=1250$

Note: A fundamental theorem in LP states that the feasible region of any LP is a convex polygon (that is the n dimensional version of two dimensional polygon), with a finite number of vertices, and further for any LP problem, there is at least one vertex which provides an optimal solution. Whenever a LP problem has more than one optimal solution, we say that there are alternative optimal solutions. Physically, this means that the resources can be combined in more than one way to maximize profit.

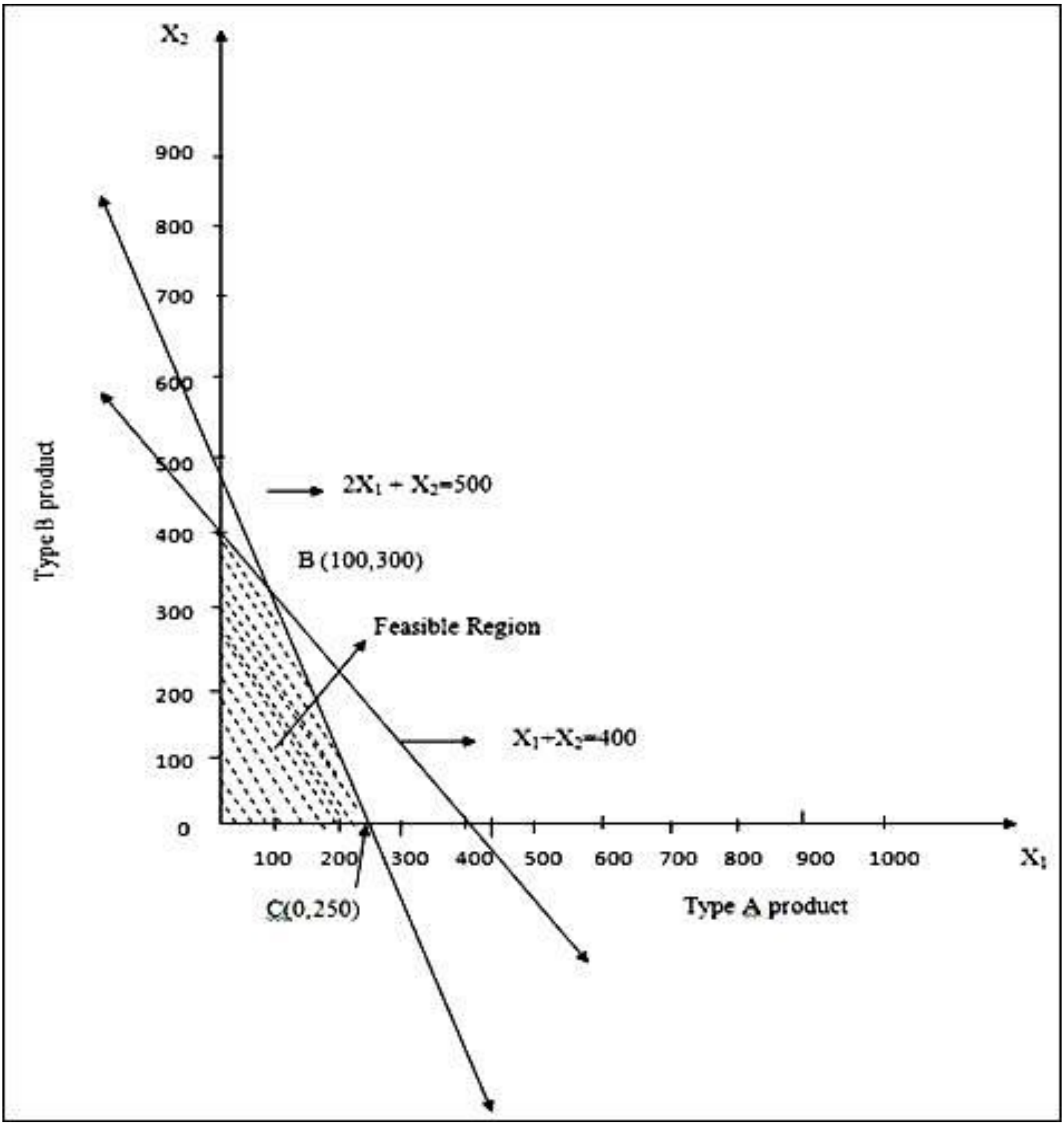

Fig. Feasible region

Example 5 Find the graphical solution of problem formulated in Example 2.

Solution

Let number of full cream and single toned milk pouches to be produced is X_1 and X_2 and Let profit be Z

Objective function: Max. $Z = 8X_1+7X_2$

Subject to constraints: $X_1 + X_2 \leq 45000$

$$X_1 \leq 20000$$

$$X_2 \leq 40000$$

$$3X_1 + X_2 \leq 66000$$

Non-negativity restrictions $X_1, X_2 \geq 0$

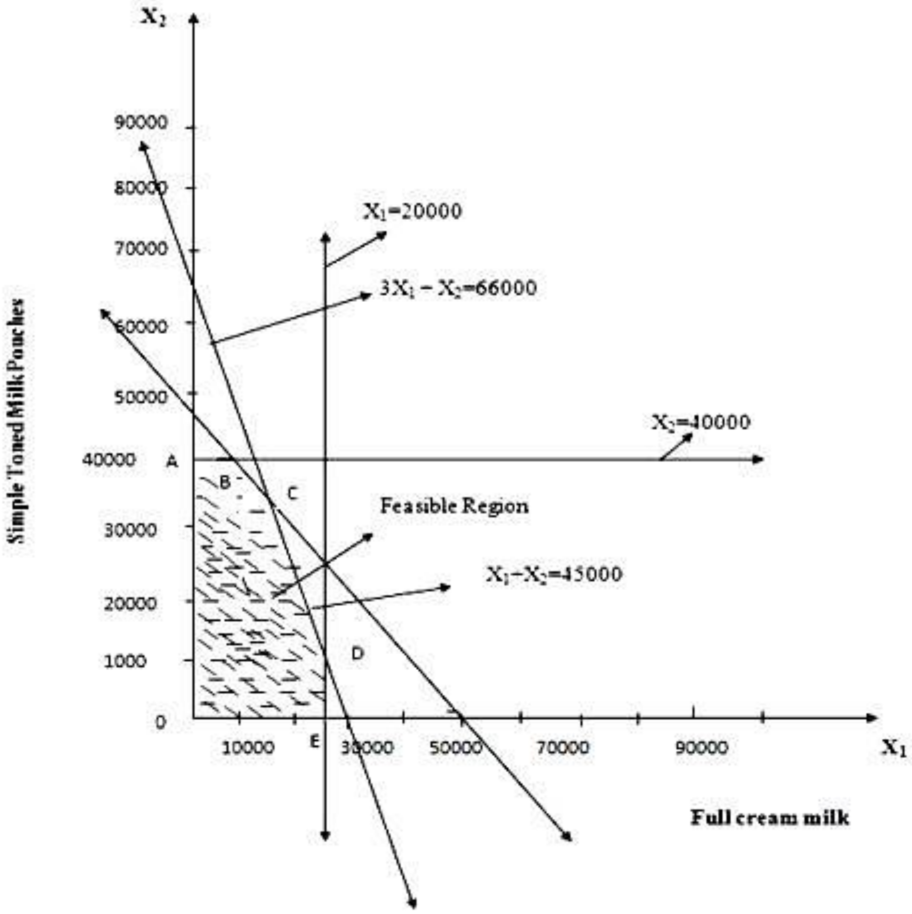

Fig. Feasible region

To find the optimal solution find the values of objective function at the various extreme points as shown in the following Table

Table Computation of maximum value of objective function

Extreme Point	Coordinates	Profit Function $Z = 8X_1+7X_2$
O	$X_1=0$, $X_2=0$	Z=8(0)+7(0)=0
A	$X_1=0$, $X_2=40000$	Z=8(0)+7(40000)=280000
B	$X_1=5000$, $X_2=40000$	Z=8(5000)+7(40000)=320000
C	$X_1=10500$, $X_2=34500$	Z=8(10500)+7(34500)=325500
D	$X_1=20000$, $X_2=6000$	Z=8(20000)+7(6000)=202000
E	$X_1=20000$, $X_2=0$	Z=8(20000)+7(0)=160000

So maximum value of Z occurs at point C (10500, 34500) so it is the optimal solution. It can be concluded that Dairy Plant must produce 10500 pouches of full cream milk and 34500 pouches of single toned milk.

Let us consider the following example to illustrate graphical solution for minimization problem

Special cases in linear programming

Up till now we have discussed those problems which have a unique optimal solution. However, it is possible for LP problem to have following special cases.

Unbounded solution

A linear programming problem is considered to have an unbounded solution if it has no limits on the constraints and further, the common feasible region is not bounded in any respect.

Example 6 Find the graphical solution of problem formulated in example 3.

Solution

Let number of units needed for food stuffs F_1 and F_2 to meet the daily requirements of vitamins A and B be respectively X_1 and X_2..

Objective function: Minimize $Z = 3X_1+2.5X_2$

Subject to constraints: $2X_1+4X_2 \geq 40$

$5X_1+2X_2 \geq 50$

Non-negativity restrictions: $X_1, X_2 \geq 0$

First of all draw the graphs of these inequalities (as discussed in example 5) .Since the inequalities are of the greater than or equal to type, the feasible region is formed by considering the area to the upper right side of each equation i.e. away from origin .The shaded area which is shown above CBD is satisfied by the two constraints as shown in Fig. 3.3 is feasible region. The feasible region for minimization problem is unbounded and unlimited. Since the optimal solution corresponds to one of the corner (extreme) points, we will calculate the values of objective function for each corner point viz. D (20, 0); B (7.5, 6.25) and C (0, 25). The calculations are shown in Table

Table showing the computation of minimum value of objective function

Extreme Point	Coordinates	Cost Function Min. $Z = 3X_1+2.5X_2$
D	$X_1=20$, $X_2=0$	$Z=3(20)+2.5(0)=60$
B	$X_1=7.5$, $X_2=6.25$	$Z=3(7.5)+2.5(6.25)=38.125$
C	$X_1=0$, $X_2=25$	$Z=3(0)+2.5(25)=62.5$

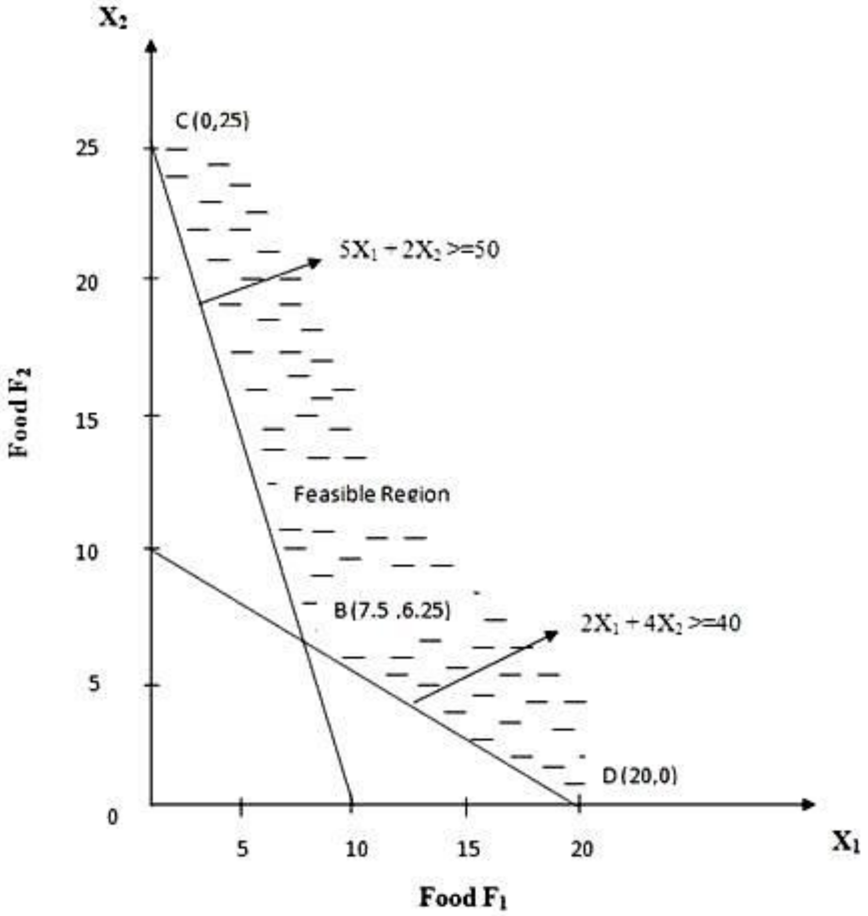

Fig. Feasible region for minimization problem

The minimum cost is obtained at the corner point B(7.5,6.25) i.e. X_1=7.5 and X_2=6.25. Hence to minimize the cost and to meet the daily requirements of vitamin A and B, number of units needed of food stuffs F_1 and F_2 be 7.5 and 6.25 respectively.

Infeasible solution

In this there is no solution to an LP Problem that satisfies all the constraints. Graphically, it means that no feasible solution region exists. Such a condition indicates that the LP problem has been wrongly formulated.

Redundant constraint

In a properly formulated LP problem, each of the constraints will define a portion of the boundary of the feasible solution region. Whenever, a constraint does not define a portion of the boundary of the feasible solution region, it is called a redundant constraint. Let us consider the following example to illustrate this.

Example 7 Maximize $Z=1170X_1+1110X_2$

Subject to:

$9X_1+5X_2 \geq 500$

$7X_1+9X_2 \geq 300$

$5X_1+3X_2 \leq 1500$

$7X_1+9X_2 \leq 1900$

$2X_1+4X_2 \leq 1000$

$X_1, X_2 \geq 0$

The feasible region of this LP problem is indicated in Fig. In this the feasible region has been formulated by two constraints.

$9X1+5X2 \geq 500$

$7X1+9X2 \leq 1900$

$X1, X2 \geq 0$

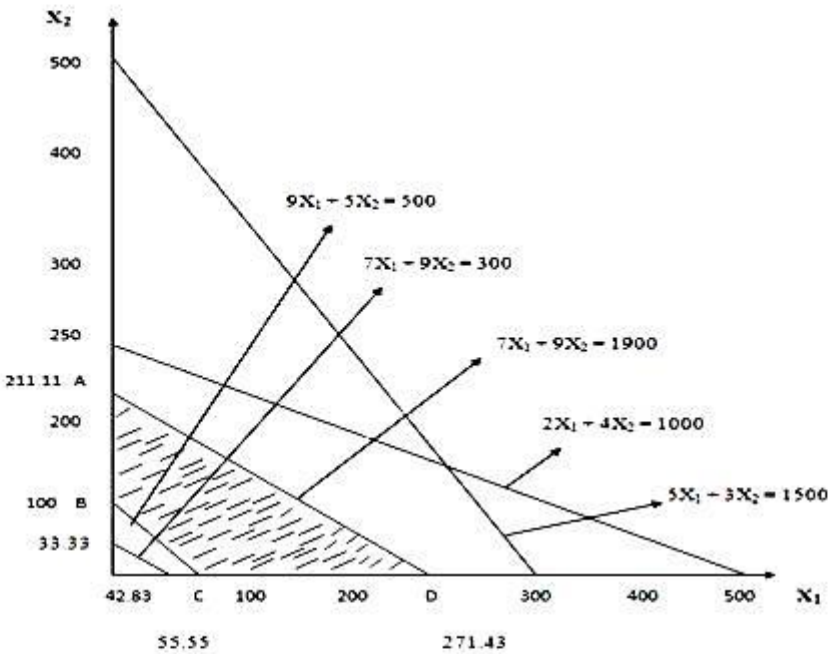

Fig Feasible Region and Redundant Constraints

The remaining three constraints, although present, is not affecting the feasible region in any manner. Such constraints are known as redundant constraints.

Solution of LPP by graphical method

After formulating the linear programming problem, our aim is to determine the values of decision variables to find the optimum (maximum or minimum) value of the objective function. Linear programming problems which involve only two variables can be solved by graphical method. If the problem has three or more variables, the graphical method is impractical.

The major steps involved in this method are as follows

(i) State the problem mathematically

(ii) Write all the constraints in the form of equations and draw the graph

(iii) Find the feasible region

(iv) Find the coordinates of each vertex (corner points) of the feasible region. The coordinates of the vertex can be obtained either by inspection or by solving the two equations of the lines intersecting at the point

(v) By substituting these corner points in the objective function we can get the values of the objective function

(vi) If the problem is maximization then the maximum of the above values is the optimum value. If the problem is minimization then the minimum of the above values is the optimum value

Example 1

Solve the following LPP

Maximize $Z = 2\,x_1 + 5x_2$

subject to the conditions $x_1 + 4x_2 \leq 24$

$3x_1 + x_2 \leq 21$

$x_1 + x_2 \leq 9$and $x_1, x_2 \geq 0$

Solution:

First we have to find the feasible region using the given conditions.

Since both the decision variables x_1 and x_2 are non-negative ,the solution lies in the first quadrant.

Write all the inequalities of the constraints in the form of equations.

Therefore we have the lines $x_1 + 4x_2 = 24$; $3x_1 + x_2 = 21$; $x_1 + x_2 = 9$ $x_1 + 4x_2 = 24$ is a line passing through the points (0 , 6) and (24 , 0). [(0,6) is obtained by taking $x_1 = 0$ in $x_1 + 4x_2 = 24$, (24 , 0) is obtained by taking $x_2 = 0$ in $x_1 + 4x_2 = 24$].

Any point lying on or below the line $x_1 + 4x_2 = 24$ satisfies the constraint $x_1 + 4x_2 \leq 24$.

$3x_1 + x_2 = 21$ is a line passing through the points (0, 21) and (7, 0). Any point lying on or below the line $3\,x_1 + x_2 = 21$ satisfies the constraint $3\,x_1 + x_2 \leq 21$.

$x_1 + x_2 = 9$ is a line passing through the points (0 , 9) and (9 , 0) .Any point lying on or below the line $x_1 + x_2 = 9$ satisfies the constraint $x_1 + x_2 \leq 9$.

Now we draw the graph.

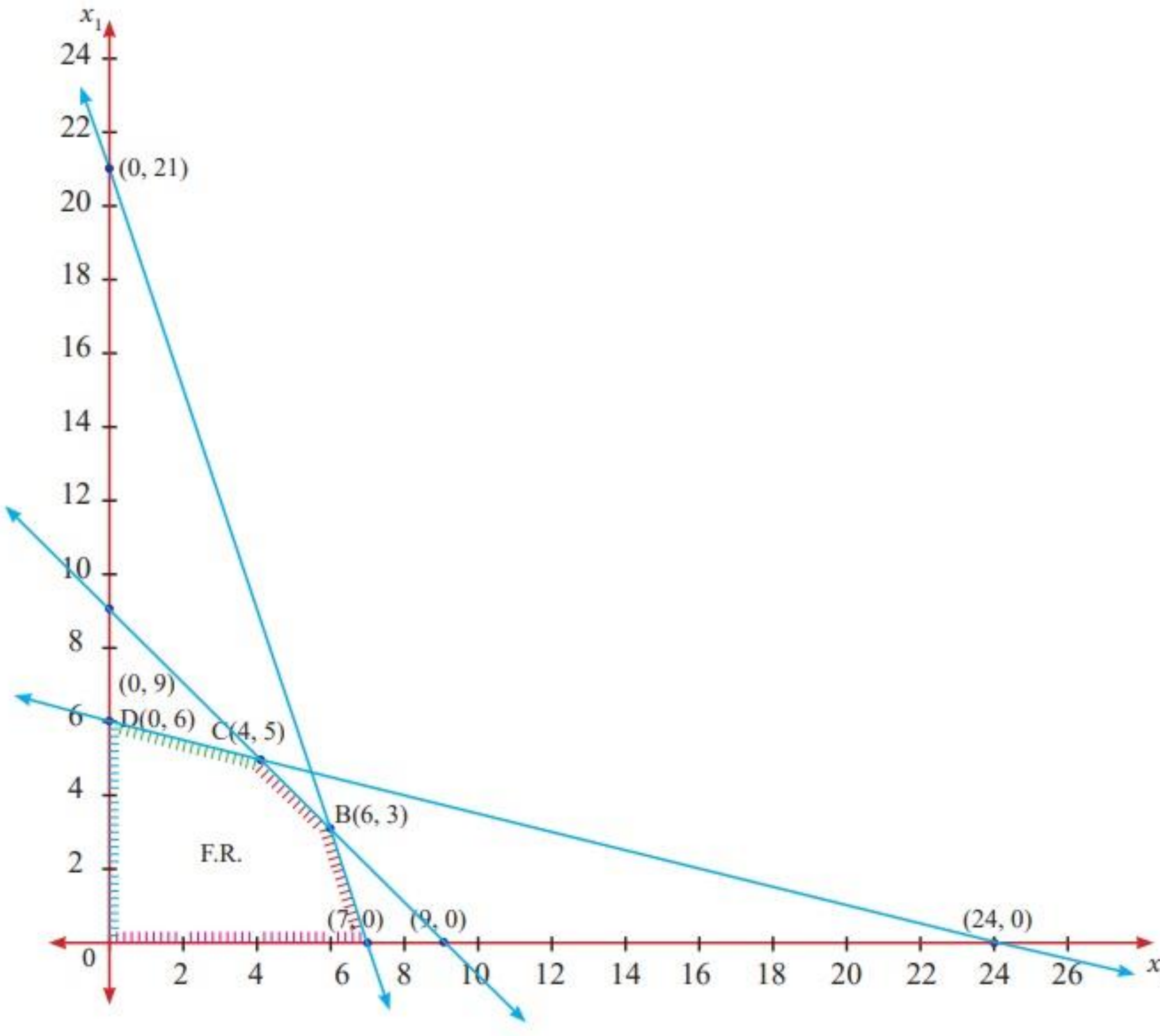

Fig 10.1

The feasible region satisfying all the conditions is OABCD.The co-ordinates of the points are O(0,0) A(7,0);B(6,3) [the point B is the intersection of two lines $x_1 + x_2 = 9$ and $3x_1 + x_2 = 21$];C(4,5) [the point C is the intersection of two lines

$x_1 + x_2 = 9$ and $x_1 + 4x_2 = 24$] and D(0,6).

Corner points	$Z = 2x_1 + 5x_2$
O(0,0)	0
A(7,0)	14
B(6,3)	27
C(4,5)	33
D(0,6)	30

Table 10.2

Maximum value of Z occurs at C. Therefore the solution is $x_1 = 4$, $x_2 = 5$, $Z_{max} = 33$

Example 2

Solve the following LPP by graphical method Minimize $z = 5x_1 + 4x_2$ Subject to constraints $4x_1 + x_2 \geq 40$; $2x_1 + 3x_2 \geq 90$ and $x_1, x_2 \geq 0$

Solution:

Since both the decision variables x_1 and x_2 are non-negative, the solution lies in the first quadrant of the plane.

Consider the equations $4x_1 + x_2 = 40$ and $2x_1 + 3x_2 = 90$

$4x_1 + x_2 = 40$ is a line passing through the points (0,40) and (10,0).Any point lying on or above the line $4x_1 + x_2 = 40$ satisfies the constraint $4x_1 + x_2 \geq 40$.

$2x_1 + 3x_2 = 90$ is a line passing through the points (0,30) and (45,0). Any point lying on or above the line $2x_1 + 3x_2 = 90$ satisfies the constraint $2x_1 + 3x_2 \geq 90$.

Draw the graph using the given constraints.

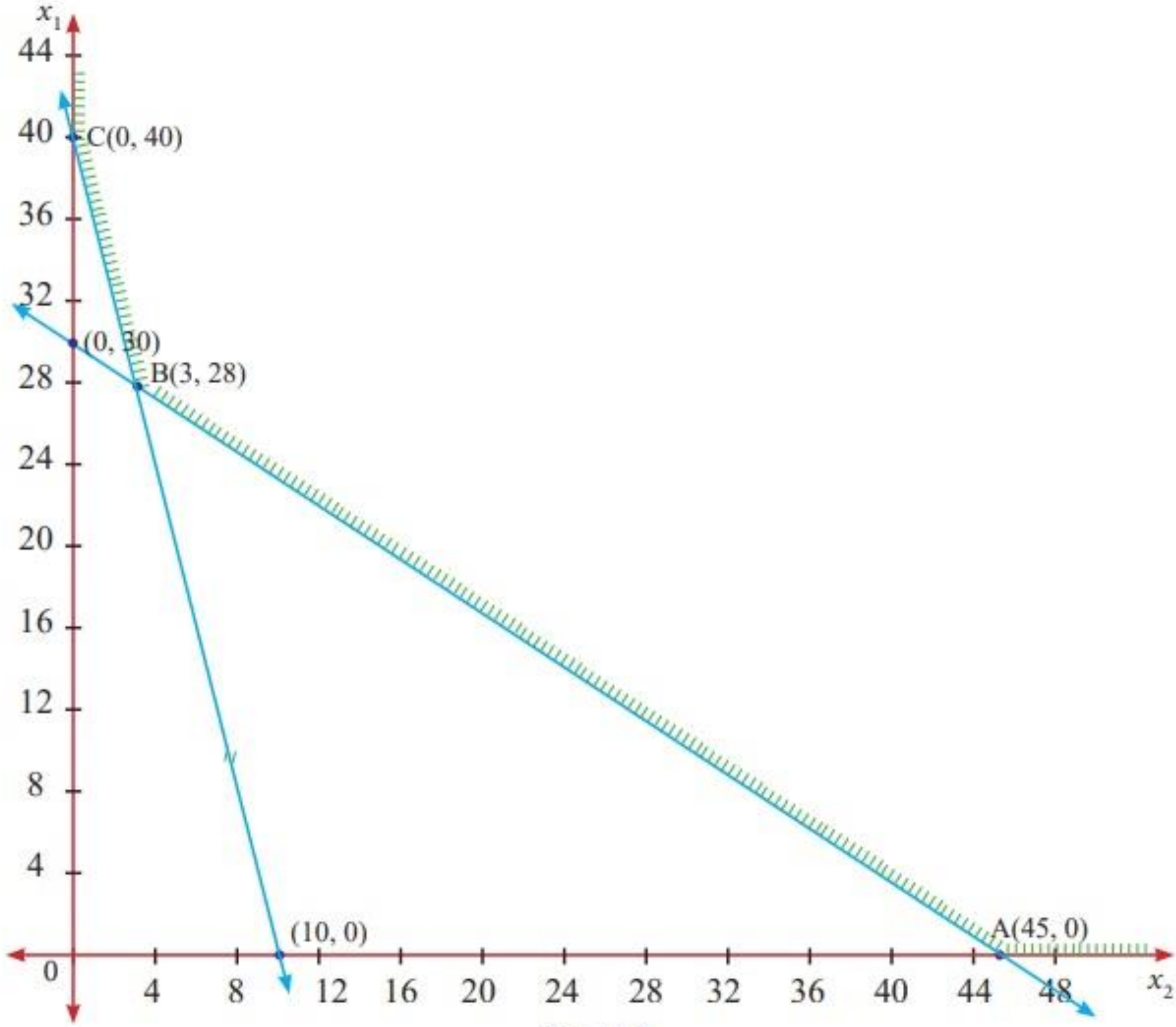

Fig 10.2

The feasible region is ABC (since the problem is of minimization type we are moving towards the origin.

Corner points	$z = 5x_1+4x_2$
A(45,0)	225
B(3,28)	127
C(0,40)	160

Table 10.4

The minimum value of Z occurs at B(3,28).

Hence the optimal solution is $x_1 = 3$, $x_2 = 28$ and $Z_{min}=127$

Example 3

Solve the following LPP.

Maximize $Z= 2x_1 + 3x_2$ subject to constraints $x_1 + x_2 \leq 30$; $x_2 \leq 12$; $x_1 \leq 20$ and $x_1, x_2 \geq 0$

Solution:

We find the feasible region using the given conditions.

Since both the decision variables x_1 and x_2 are non-negative, the solution lies in the first quadrant of the plane.

Write all the inequalities of the constraints in the form of equations.

Therefore we have the lines

$x_1+x_2=30$; $x_2 =12$; $x_1= 20$

$x_1+x_2 =30$ is a line passing through the points (0,30) and (30,0)

$x_2 = 12$ is a line parallel to x_1–axis

$x_1 = 20$ is a line parallel to x_2–axis.

The feasible region satisfying all the conditions $x_1+ x_2 \leq 30$; $x_2 \leq 12$; $x_1 \leq 20$ and $x_1, x_2 \geq 0$ is shown in the following graph.

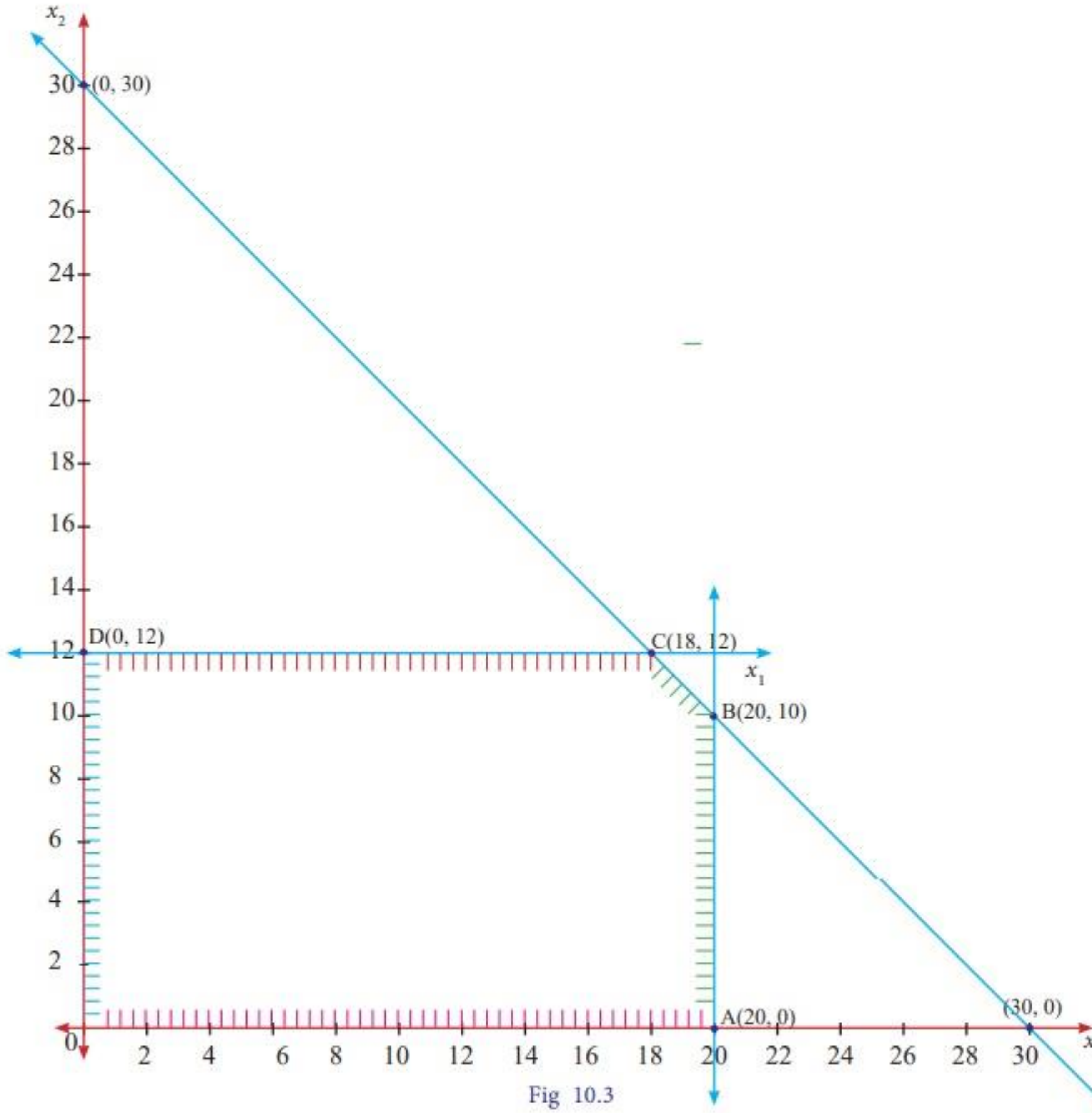

Fig 10.3

The feasible region satisfying all the conditions is OABCD.

The co-ordinates of the points are O(0,0) ; A(20,0); B(20,10) ; C(18,12) and D(0,12).

Corner points	$Z = 2x_1 + 3x_2$
O(0,0)	0
A(20,0)	40
B(20,10)	70
C(18,12)	72
D(0,12)	36

Table 10.3

Maximum value of Z occurs at C. Therefore the solution is $x_1 = 18$, $x_2 = 12$, $Z_{max} = 72$

Example 4

Maximize $Z = 3x_1 + 4x_2$ subject to $x_1 - x_2 \leq -1$; $-x_1 + x_2 \leq 0$ and $x_1, x_2 \geq 0$

Solution:

Since both the decision variables x_1, x_2 are non-negative ,the solution lies in the first quadrant of the plane.

Consider the equations $x_1 - x_2 = -1$ and $-x_1 + x_2 = 0$

$x_1 - x_2 = -1$ is a line passing through the points (0,1) and (–1,0)

$-x_1 + x_2 = 0$ is a line passing through the point (0,0)

Now we draw the graph satisfying the conditions $x_1 - x_2 \leq -1$; $-x_1 + x_2 \leq 0$ and $x_1, x_2 \geq 0$

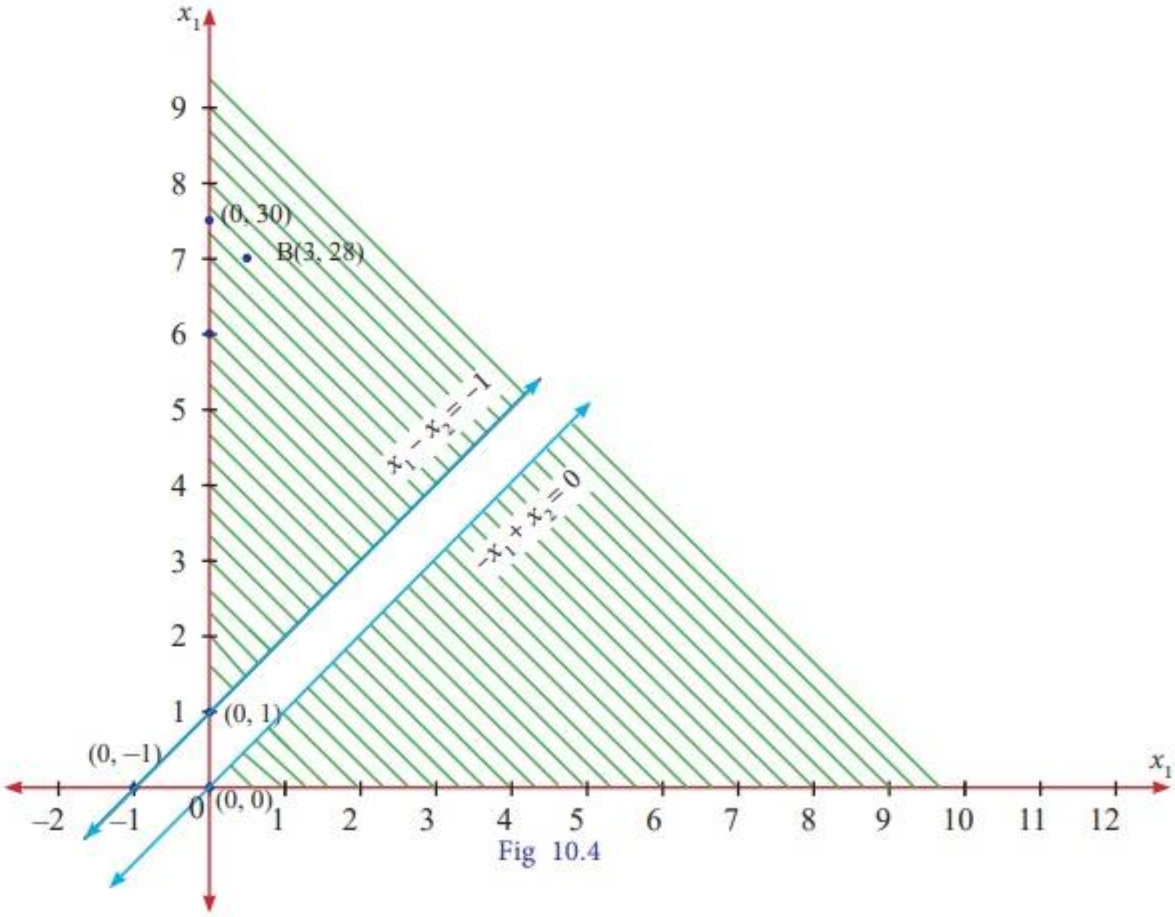

Fig 10.4

There is no common region(feasible region) satisfying all the given conditions.

Hence the given LPP has no solution.

Exercise 1

1. A company produces two types of pens A and B. Pen A is of superior quality and pen B is of lower quality . Profits on pens A and B are Rs 5 and Rs 3 per pen respectively. Raw materials required for each pen A is twice as that of pen B. The supply of raw material is sufficient only for 1000 pens per day . Pen A requires a special clip and only 400 such clips are available per day. For pen B, only 700 clips are available per day . Formulate this problem as a linear programming problem.

Pen A-X

Pen B-Y

Max Z= 5X+3Y

2X+Y<=1000

X<=400

Y<=700

X,Y>=0

2. A company produces two types of products say type A and B. Profits on the two types of product are Rs.30/- and Rs.40/- per kg respectively. The data on resources required and availability of resources are given below.

	Requirements		Capacity available per month
	Product A	Product B	
Raw material (kgs)	60	120	12000
Machining hours / piece	8	5	600
Assembling (man hours)	3	4	500

Formulate this problem as a linear programming problem to maximize the profit.

Max Z= 30X+40Y

Subject to the constraints

60 X+120Y<=12000

8X+5Y<=600

3X+4Y<=500

X,Y>=0

3. A company manufactures two models of voltage stabilizers viz., ordinary and auto-cut. All components of the stabilizers are purchased from outside sources , assembly and testing is carried out at company's own works. The assembly and testing time required for the two models are 0.8 hour each for ordinary and 1.20 hours each for auto-cut. Manufacturing capacity 720 hours at present is available per week. The market for the two models has been surveyed which suggests

maximum weekly sale of 600 units of ordinary and 400 units of auto-cut . Profit per unit for ordinary and auto-cut models has been estimated at Rs 100 and Rs 150 respectively. Formulate the linear programming problem.

	Ordinary	Auto cut	
Assembly time	0.8	1.2	360
Testing time	0.8	1.2	360
Profit	100	150	

4. Solve the following linear programming problems by graphical method.

(i) Maximize Z = $6x_1 + 8x_2$ subject to constraints $30x_1+20x_2 \leq 300$;$5x_1+10x_2 \leq 110$; and x_1, $x_2 \geq 0$.

(ii) Maximize $Z = 22x_1 + 18x_2$ subject to constraints $960x_1 + 640x_2 \leq 15360$; $x_1 + x_2 \leq 20$ and x_1 , $x_2 \geq 0$.

(iii) Minimize $Z = 3x_1 + 2x_2$ subject to the constraints $5x_1+ x_2 \geq 10$; $x_1+ x_2 \geq 6$; $x_1+ 4x_2 \geq 12$ and x_1, $x_2 \geq 0$.

(iv) Maximize $Z = 40x1 + 50x2$ subject to constraints $30x1 + x2 \leq 9$; $x1 + 2x2 \leq 8$ and $x1$, $x2 \geq 0$

(v) Maximize $Z = 20x_1 + 30x_2$ subject to constraints $3x_1 + 3x_2 \leq 36$; $5x_1 + 2x_2 \leq 50$; $2x_1 + 6x_2 \leq 60$ and x_1 , $x_2 \geq 0$

(vi) Minimize $Z = 20x_1 + 40x_2$ subject to the constraints $36x_1 + 6x_2 \geq 108$, $3x_1 + 12x_2 \geq 36$, $20x_1 + 10x_2 \geq 100$ and x_1 , $x_2 \geq 0$

Introduction

In the previous chapter we considered the formulation of linear programming problems and the graphic method of solving them. Although the graphical approach to the solution of such problems is an invaluable aid to understand its basic structure, the method is of limited application in industrial problems as the number of variables occurring there is often considerably large. The Simplex Method provides an efficient technique which can be applied for solving linear programming problems of any magnitude-involving two or more decision variables. It is an iterative procedure having fixed computational rules that leads to a solution to the problem in a finite number of steps.

Principle of Simplex Method

As the fundamental theorem of LP problem tells us that at least one basic feasible solution of any LP problem must be optimal, provided the optimal solution of the LP problem exists. Also, the number of basic feasible solutions of the LP problem is finite and at the most ${}^{n}C_{m}$ (where, n is number of decision variables and m is the number of constraints in the problem). On the other hand, the feasible solution may be infinite in number. So it is rather impossible to search for optimal solutions from amongst all feasible solutions. Furthermore, a great labour will also be required in finding out all the basic feasible solutions and select that one which optimizes the objective function. In order to remove this difficulty; a simple method was developed by Dantzig (1947) which is known as Simplex Algorithm. Simplex Algorithm is a systematic and efficient procedure for finding corner point solutions and taking them for optimality. The evaluation of corner points always starts from the point of origin. This solution is then tested for optimality i.e. it tests whether an improvement in the objective function is possible by moving to adjacent corner point of the feasible function space. This iterative search for a better corner point is repeated until an optimal solution if it exists, is determined.

Basic Terms Involved in Simplex Procedure

The following terms relevant for solving a linear programming problem through simplex pro cedure are given below:

Standard form of linear programming problem

The standard form of LP problem is to develop the procedure for solving general LP problem. The optimal solution of the standard form of a LP problem is the same as original LP problem. The characteristics of standard form are given in following steps:

- **Step 1.** All the constraints should be converted to equations except for the non-negativity restrictions which remain as inequalities (≥0).
- **Step 2.** The right side element of each constraint should be made non-negative.
- **Step 3.** All variables must have non-negative values.
- **Step 4.** The objective function should be of maximization form.

Slack variables

If a constraint has less than or equal sign, then in order to make it on equality we have to add something positive to the left hand side. The non-negative variable which is added to the left hand side of the constraint to convert it into equation is called the slack variable. For example, consider the constraints.

$3X_1 + 5X_2 \leq 2$, $7X_1 + 4X_2 \leq 5$, $X_1, X_2 \geq 0$

We add the slack variables $S_1 \geq 0$, $S_2 \geq 0$ on the left hand sides of above inequalities respectively to obtain

$3X_1+5X_2+S_1 = 2$

$7X_1+4X_2+S_2 = 5$

$X_1, X_2, S_1, S_2 \geq 0$

Surplus variables

If a constraint has greater than or equal to sign, then in order to make it an equality we have to subtract something non-negative from its left hand side. The positive variable which is subtracted from the left hand side of the constraint to convert it into equation is called the surplus variable.

For example, consider the constraints.

$3X_1 + 5X_2 \geq 2$, $2X_1 + 4X_2 \geq 5$, $X_1, X_2 \geq 0$

We subtract the surplus variables $S_3 \geq 0$, $S_4 \geq 0$ on the left hand sides of above inequalities respectively to obtain

3X1+5X2 -S3 = 2

2X1 + 4X2 -S4 = 5

X1, X2, X3, X4 ≥0

Solution to LPP

Any set $X = \{X_1, X_2, X_3,\ldots, X_{n+m}\}$ of variables is called a solution to LP problem, if it satisfies the set of constraints only.

Feasible solution (FS)

Any set $X = \{X_1, X_2, X_3,\ldots,X_{n+m}\}$ of variables is called a feasible solution of L.P. problem, if it satisfies the set of constraints as well as non-negativity restrictions.

Basic solution (BS)

For a system of m simultaneous linear equations in n variables (n>m), a solution obtained by setting (n-m) variables equal to zero and solving for the remaining variables is called a basic solution. Such m variables (of course, some of them may be zero) are called basic variables and remaining (n-m) zero-valued variables are called non-basic variables.

Basic feasible solution (BFS)

A basic feasible solution is a basic solution which also satisfies the non-negativity restrictions, that is all basic variables are non-negative. Basic solutions are of two types

(a) **Non-degenerate BFS**: A non-degenerate basic feasible solution is the basic feasible solution which has exactly m positive X_j (j=1, 2,…,m). In other words, all basic m variables are positive, and the remaining (n - m) variables will be all zero.

(b) **Degenerate BFS**: A basic feasible solution is called degenerate, if one or more basic variables are zero-valued.

Optimum basic feasible solution

A basic feasible solution is said to be optimum, if it also optimizes (maximizes or minimizes) the objective function.

Unbound solution

If the value of the objective function Z can be increased or decreased indefinitely, such solutions are called unbounded solutions.

Computational Aspect of Simplex Method for Maximization Problem

Step 1: Formulate the linear programming model. If we have n-decision variables X_1, X_2, X_n and m constraints in the problem , then mathematical formulation of L P problem is
Maximize $Z=C_1X_1+ C_2X_2+....+C_nX_n$

Subject to the constraints:

$a_{11} X_1+ a_{12} X_2+ .... +a_{1n} X_n \leq b_1$
$a_{21} X_1+ a_{22} X_2 + ... + a_{2n} X_n \leq b_2$
$a_{m1} X_1+ a_{m2} X_2+.... + a_{mn} X_n \leq b_m$
$X_1, X_2, \dots X_n \geq 0$

Step 2: Express the mathematical model of LP problem in the standard form by adding slack variables in the left hand side of the constraints and assign zero coefficient to these variables in the objective function.Thus we can restate the problem in terms of equations as follows:

Maximize $Z=C_1X_1+ C_2X_2 ++ C_nX_n +0X_{n+1} +0X_{n+2}+........+0X_{n+m}$
Subject to the constraints:

$$a_{11} X_1+ a_{12} X_2+ ....+ a_{1n} X_n +X_{n+1} = b_1$$
$$a_{21} X_1+ a_{22} X_2+....+ a_{2n} X_n +X_{n+2} = b_2$$
$$.$$
$$a_{m1} X_1+ a_{m2}X_2+....+a_{mn} X_n +X_{n+m} = b_m$$

$X_1, X_2,....... ,X_{n,}X_{n+1},X_{n+2} ,, X_{n+m} \geq 0$

Step 3: Design the initial feasible solution .An initial basic feasible solution is obtained by setting $X_1=X_2=.... =X_n =0$. Thus, we get $X_{n+1}=b_1$, $X_{n+2} =b_2$,......, $X_{n+m} =b_m$.
Step 4: Construct the starting (initial) simplex tableau. For computational efficiency and simplicity, the initial basic feasible solution, the constraints of the standard LP problem as well as the objective function can be displayed in a tabular form, called the *Simplex Tableau* as shown below.

Table 1 Initial simplex tableau

C_j (contribution per unit) →			c_1	c_2		c_n	0	0		0	Minimum Ratio*
C_b	Basic	Value of	Coefficient Matrix				Identify Matrix				

	Variables	Basic Variables									
	B	b(=X_B)	X_1	X_2		X_n	X_{n+1}	X_{n+2}		X_{n+m}	
0	s_1	b_1	a_{11}	a_{12}		a_{1n}	1	0		0	
0	s_2	b_2	a_{21}	a_{22}		a_{2n}	0	1		0	
.	.	.	.	.		.					
.	.	.	.	.		.					
.	.	.	.	.		.					
0	s_m	b_m	a_{m1}	a_{m2}		a_{mn}	0	0		1	
Contribution loss per unit:		$Z = \sum C_{Bj} a_{ij}$	0	0		0	0	0		0	
Net contribution per units:		$\Delta_j = Z_j - C_j$	c_1	c_2		c_n	0	0		0	

* Negative ratio is not to be considered.

The interpretation of the data in the above *Tableau* is given as under. Other simplex tableau will have similar interpretations.

(i) The first row, called the *objective row* of the simplex table indicates the values of C_j (j subscript refer to the column number) which are the coefficients of the (m + n) variables in the objective function. These coefficients are obtained directly from the objective function and the value C_j would remain the same in the succeeding tables. The second row of the table provides the major column headings for the table and these column headings remain unchanged in the succeeding tables of the Simplex Method.

(ii) The first column labelled C_B, also known as *objective column*, lists the coefficient of the current basic variables in the objective function. The second column labelled *Basic variables* points out the basic variables in the basis, and in the initial simplex tableau these basic variables are the slack variables. The third column labelled *Solution values* (= x_B), indicates the resources or the solution values of the basic variables.

(iii) The *body matrix* (under non-basic variables) in the initial simplex tableau consists of the coefficients of the decision variables in the constraint set.

(iv)The *identity matrix* in the initial simplex tableau represents the coefficient of the slack variables that have been added to the original inequalities to make them equation. The matrix under non-basic variables in the simplex tableau is called *coefficient matrix.* Each simplex tableau contains an identity matrix under the basic variables.

(v) To find an entry in the Z_j row under a column, we multiply the entries of that column by the corresponding entries of C_B column and add the products, *i.e.,* Z = $\sum C_B a_{ij}$.

The Z_j entry under the *Solution column* gives the current value of the objective function.

(vi) The final row labeled $\Delta_j = Z_j - C_j$ called the *index* (or net evaluation) row, is used to determine whether or not the current solution is optimal. The calculation of $Z_j - C_j$ row simply involves subtracting each Z_j value from the corresponding C_j value for that column, which is written at the top of that column. Each entry in the Δ_j row represents the net contribution (or net marginal improvement) to the objective function that results by introducing one unit of each of the respective column variables.

Step 5: *Test the Solution for Optimality*. Examine the index row of the above simplex tableau. If all the elements in the index row are positive then the current solution is optimal. If there exist some negative values, the current solution can further be improved by removing one basic variable from the basis and replacing it by some non-basic one.

Step 6: *Revision of the Current Simplex Tableau*. At each iteration, the Simplex Method moves from the current basic feasible solution to a better basic feasible solution. This involves replacing one current basic variable (called the departing variable) by a new non-basic variable (called the entering variable).

(a) *Determine which variable to enter into the solution-mix net.* One way of doing this is by identifying the column (and hence the variable) with the most negative number in the Δ_j row of the previous table.

(b) *Determine the departing variable or variable to be replaced.* Next we proceed to determine which variable must be removed from the basis to pave way for the entering variable. This is accomplished by dividing each number in the quantity (or solution values) column by the corresponding number in the pivot column selected in (a), *i.e.,* we compute the respective ratios b_1/a_{1j}, b_2/a_{2j}, …, b_m/a_{mj} (only for those a_{ij}s; i=1,2,…., m which are strictly positive). These quotients are written in the last column labelled Minimum Ratio of the simplex tableau. The row corresponding to smallest of these non-negative ratios is called the pivot (or key) row and the corresponding basic variable will leave the basis. Let the minimum of $\{ b_1/a_{1j}, b_2/a_{2j}, …., b_m/a_{mj} ; a_{ij} > 0\}$ be b_k/a_{kj}, then corresponding variables in the pivot row s_k will be termed as outgoing (or departing) variable in the next tableau to be constructed just after we put an arrow → of type to right of k^{th} row of the simplex tableau 1.

(c) *Identify the pivot number.* The non-zero positive number that lies at the intersection of the pivot column and pivot row of the given table is called the pivot (or key) number. We place a circle around the number.

Step 7: *Evaluate (update) the new solution.* After identifying the entering and departing variable, find the new basic feasible solution by constructing a new simplex tableau from the current one by using the following steps:

(a) Compute new values for the pivot row by simply dividing every element of the pivot row by the pivot number.

(b) New entries in the C_B column and X_B column are entered in the new table of the current solution

(c) Compute new values for each of the remaining rows by using the following formula

New row numbers=Number in old rows-{(corresponding number above or below pivot number) x (corresponding number in the row replaced in (a))}

(d) Test for optimality. Compute the Z_j and index rows as previously demonstrated in the initial simplex tableau. If all numbers in the index row are either zero or positive, an optimal solution has been made attained. *i.e.,* there is no variable which can be introduced in the solution to cause the objective function value to increase.

4. *Revise the solution.* If any of the numbers in the index ($\Delta_j = Z_j - C_j$) row are negative, repeat the entire steps 5 & 6 again until an optimal solution has been obtained.

The above procedure is illustrated through the following example.

Example 1

A firm produces three products A, B, and C each of which passes through three different departments fabrication, finishing, packaging. Each unit of product A requires 3, 4 and 2 hours respectively, B requires 5, 4 and 4 hours respectively and C requires 2, 4 and 5 hours respectively in 3 departments respectively. Every day 60 hours are available in fabrication department, 72 hours in finishing and 100 hours in packaging department. If unit contribution of unit A is Rs. 5, Rs. 10 for B and Rs. 3 for C. Then determine number of units of each product so that total contribution to cost is maximized and also determine if any capacity would remain unutilized.

Solution:

Step 1: Formulate this as LPP. Let X_1, X_2 and X_3 be the number of units produced of the products A, B and C respectively.

Objective function: $\text{Max } Z = 5X_1 + 10X_2 + 3X_3$

Subject to constraints: $3X_1 + 5X_2 + 2X_3 \le 60$

$4X_1 + 4X_2 + 4X_3 \le 72$

$2X_1 + 4X_2 + 5X_3 \le 100$ $\quad X_1, X_2, X_3 \ge 0$

Step 2: Now converting into standard form of LPP

$$\text{Max } Z = 5X_1 + 10X_2 + 3X_3 + 0S_1 + 0S_2 + 0S_3$$

$$3X_1 + 5X_2 + 2X_3 + S_1 = 60$$

$$4X_1 + 4X_2 + 4X_3 + S_2 = 72$$

$$2X_1 + 4X_2 + 5X_3 + S_3 = 100 \qquad X_1, X_2, X_3, S_1, S_2, S_3 \ge 0$$

where S_1, S_2 and S_3 are slack variables.

Step 3: Find the initial feasible solution .An initial basic feasible solution is obtained by setting $X_1 = 0$, $X_2 = 0$ and $X_3 = 0$. Thus, we get $S_1 = 60$, $S_2 = 72$ and $S_3 = 100$.

Step 4: Construct the starting (initial) simplex tableau.

			Cj →	5	10	8	0	0	0	
	B.V.	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	Minimum Ratio
R_1	S_1	0	60	3	**(5)**	2	1	0	0	60/5=12 →
R_2	S_2	0	72	4	4	4	0	1	0	72/4=18
R_3	S_3	0	100	2	4	5	0	0	1	100/4=25
			Z_j	0	0	0	0	0	0	
			Δ_j	−5	−10	−8	0	0	0	

Step 5: The most negative value of Δ_j is −10 hence X_2 is the incoming variable (↑) and the least positive minimum ratio is 12 hence S_1 is the outgoing variable (⟶). The element under column X_2 and row R_1 is the key element i.e. 5 so divide each element of row R_1 by 5 (i.e. $R_1^a \to \frac{R_1}{5}$). Subtract appropriate multiples of this new row from the remaining rows, so as to obtain zeros in the remaining positions. Performing the row operations $R_2^b - R_2^a \to 4R_1^a \to and\ R_3^b - R_3^a \to 4R_1^a$

we get the second **Simplex tableau as**

			Cj →	5	10	8	0	0	0	
	B.V.	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	M.R.
R_1^b	X_2	10	12	3/5	1	2/5	1/5	0	0	12/2/5=30
R_2^b	S_2	0	24	8/5	0	12/5	−4/5	1	0	24/12/5=10⟶
R_3^b	S_3	0	52	-2/5	0	17/5	−4/5	0	1	52/17/5=15.294
			Z_j	6	10	4	2	0	0	
			Δ_j	1	0	−4	2	0	0	

Step 6: The most negative value of Δ_j is -4 hence X_3 is the incoming variable (↑) and the least positive minimum ratio is 10 hence S_2 is the outgoing variable (�). The element under column X_2 and row

R_1 is the key element i.e. 5 so divide each element of row R_2^b by 12/5 (i.e. $R^c \to R^b * 5/12$). Subtract appropriate multiples of this new row from the remaining rows, so as to obtain zeros in the remaining positions. Performing the row operations $R_1^d \to R_1^c - \frac{2}{5}R_2^c$ and $R_3^d \to R_3^c - \frac{17}{5}R_2^c$.

We get the third **Simplex tableau as**

		Cj →	5	10	8	0	0	0
B.V.	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3
X_2	10	8	1/3	1	0	1/3	−1/6	0
X_3	8	10	2/3	0	1	−1/3	5/12	0
S_3	0	18	−8/3	0	0	1/3	−17/12	1
		Z_j	26/3	10	8	2/3	5/3	0
		Δ_j	11/3	0	0	2/3	5/3	0

It is apparent from this table that all $\Delta_j = Z_j - C_j$ are positive and therefore an optimum solutions is reached . So $X_1 = 0$, $X_2 = 8$, $X_3 = 10$

$Z = 5X_1 + 10X_2 + 8X_3 = 160$

And also as S_3 is coming out to be 18 so there are 18 hours unutilized in finishing department. In case the objective function of the given LP problem is to be minimized, then we convert it into a problem of maximization by using
Min. $Z^* = -$ Max. $(-Z)$. The procedure of finding optimal solution using Simplex Method is illustrated through the following example:

Example 2

Minimize the objective function Z: $X_1 - 3X_2 + 2X_3$
Subject to constraints $3X_1 - X_2 + 3X_3 \le 7$

$$-2X_1 + 4X_2 \le 12$$

$$-4X_1 + 3X_2 + 8X_3 \le 10$$

$$X_1, X_2, X_3 \ge 0$$

Solution

Converting this minimization problem into maximization problem

Objective function Max $Z^{\#}$: $-X_1 + 3X_2 - 2X_3 + 0S_1 + 0S_2 + 0S_3$

Constraints

$3X_1 - X_2 + 3X_3 + S_1 = 7$

$-2X_1 + 4X_2 + S_2 = 12$

$-4X_1 + 3X_2 + 8X_3 + S_3 = 10$

$X_1, X_2, X_3, S_1, S_2, S_3 \ge 0$

Starting Simplex Table

		Cj	−1	3	−2	0	0	0	
B.V.	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	Min. Ratio
S_1	0	7	3	−1	3	1	0	0	-
S_2	0	12	−2	4	1	0	1	0	3 →

S_3	0	10	−4	3	8	0	0	1	10/3
		Δ_j	1	−3	2	0	0	0	

Now performing the row operations $R_2 \to \frac{R_2}{4}, R_1 \to R_1 + R_2 \text{ and } R_3 \to R_3 - 3 \times R_2$

		Cj	−1	3	−2	0	0	0	Min. Ratio
B.V.	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3	
S_1	0	10	5/2	0	3	1	1/4	0	4 →
X_2	3	3	−1/2	1	0	0	1/4	0	-
X_3	0	1	−5/2	0	8	0	−3/4	1	-
		Δ_j	−1/2	0	2	0	3/4	0	

Now performing the row operations $R_1^1 \to R_1^1 \times \frac{2}{5}, R_1^1 \to R_2^1 + \frac{1}{2} \times R_1^1 \text{ and } R_3^1 \to R_3^1 + \frac{5}{2} \times R_1^1$

↑		Cj	−1	3	−2	0	0	0
B.V.	C_B	X_B	X_1	X_2	X_3	S_1	S_2	S_3
X_1	−1	4	1	0	6/5	2/5	1/10	0
X_2	3	5	0	1	3/5	1/5	3/10	0
S_3	0	11	0	0	11	1	−1/2	1
		Δ_j	0	0	13/5	1/5	4/5	0

Now all Δ_j are positive. Therefore, Optimal Solution is reached

Therefore, $X_1 = 4$, $X_2 = 5$ & $X_3 = 0$ & Max $Z^{\#} = -1 \times 4 + 3 \times 5 = 11$ Or Minimum $Z = -Z^{\#} = -11$

INITIAL BASIC FEASIBLE SOLUTION OF TRANSPORTATION PROBLEM

Introduction

In the last lesson we have learnt about Transportation Problem (TP) and its formulation. Transportation problem can be solved by simplex method and transportation method. In simplex method the solution is very lengthy and cumbersome process because of the involvement of a large number of decision and artificial variables. In this lesson we will look for an alternate solution procedure called transportation method in which initial basic feasible solution of a TP can be obtained in a better way by exploiting the special structure of the problem.

The following terms are to be defined with reference to Transportation Problem

Feasible solution (FS)

By feasible solution we mean a set of non-negative individual allocations ($X_{ij} \geq 0$) which satisfies the row and column conditions (rim requirement).

Basic feasible solution (BFS)

A feasible solution is said to be basic if the number of positive allocations equals m+n-1; that is one less than the number of rows and columns in a transportation problem.

Optimal solution

A feasible solution (not necessarily basic) is said to be optimal if it minimizes the total transportation cost.

Solution for Transportation Problem

The solution algorithm to a transportation problem can be summarized into following steps:

Step 1: Formulate the problem. The formulation of transportation problem is similar to a LP problem formulation. Here the objective function is to minimize the total transportation cost and the constraints are the supply and demand available at each source and destination, respectively.

Step 2: Obtain an initial basic feasible solution. This initial basic feasible solution can be obtained by using any of the following five methods:

a) North West Corner Rule
b) Minimum cost method
c) Vogel's Approximation Method

The solution obtained by any of the above methods must fulfill the following conditions:

i. The solution must be feasible, i.e., it must satisfy all the supply and demand constraints. This is called **rim requirement**.
ii. The number of positive allocations must be equal to m+-1, where, m is number of rows and n is number of columns.

The solution that satisfies both the above mentioned conditions is called a non-degenerate basic feasible solution.

Step 3: Test the initial solution for optimality. Using any of the following methods one can test the optimality of an initial basic solution:

i.Stepping Stone Method
ii. Modified Distribution Method **(MODI)**

If the solution is optimal then stop, otherwise, find a new improved solution.

Step 4: Updating the solution. Repeat Step 3 until the optimal solution is obtained.

Methods of Obtaining an Initial Basic Feasible Solution

Three methods are described to obtain the initial basic feasible solution of the transportation problem. These methods can be explained by considering the following example

Example 1

Let us consider the formulation of TP which can be represented by the following transportation table:

Routes	Chilling centers				Route Capacity
	P	Q	R	S	
A	16	18	21	12	150
B	17	19	14	13	160
C	32	11	15	10	90
Chilling Centre Capacity	140	120	90	50	400

Total supply and demand

Each cell in the table represents the amount transported from one route to one chilling center. The amount placed in each cell is, therefore, the value of a decision. The smaller box within each cell contains the unit transportation cost for that route.

North west corner rule (NWC) method

Step I: The first assignment is made in the cell occupying the upper left-hand (North West) corner of the transportation table. The maximum feasible amount is allocated there. That is X_{11} = Min (a_1, b_1) and this value of X_{11} is then entered in the cell (1, 1) of the transportation table.

Step II: a) If $b_1>a_1$, we move down vertically to the second row and make the second allocation of magnitude X_{21}=Min. (a_2, b_1- X_{11}) in the cell (2, 1).

b) If $b_1< a_1$ we move horizontally to the second column and make the second allocation of magnitude X_{12} = Min. (a_1-X_{11}, b_2) in the cell (1, 2).

c) If $b_1= a_1$, there is a tie for the second allocation. One can make the second allocation of magnitude X_{12} = Min. (a_1-a_1, b1) = 0 in the cell (1, 2) or X_{21}=Min. (a_2, b_1- b_1) = 0 in the cell (2, 1)

Step III: Repeat steps I & II by moving down towards the lower right corner of the transportation table until all the rim requirements are satisfied.

Illustration of north west corner rule

Let us illustrate the above method with the help of example 1given above. Following North-West corner rule, the first allocation is made in the cell (1,1), the magnitude being X_{11}=Min.(150,140)=140. The second allocation is made in cell (1,2) and the magnitude of allocation is given by X_{12}=Min. (150-140,120)=10. Third allocation is made in the cell (2,2), the magnitude being X_{22}=Min. (160,120-10)=110. The magnitude of fourth allocation in the cell (2,3) is given by X_{23}=Min. (160-110, 90)=50. The fifth allocation is made in the cell (3,3), the magnitude being X_{33}= Min. (90-50, 90)=40. The sixth allocation in the cell (3,4) is given by X_{34}=Min. (90-40, 50) = 50. Now all requirements have been satisfied and hence an initial basic feasible solution to T.P. has been obtained.

Routes	Chilling centers								Route Capacity
	P		Q		R		S		
A	140	16	10	18		21		12	150
B		17	110	19	50	14		13	160
C		32		11	40	15	50	10	90
Chilling Centre Capacity	140		120		90		50		400

The transportation cost according to the above allocation is given by z = 16 * 140 + 18 * 10 + 19 * 110 + 14 * 50 + 15 * 40 + 10 * 50 = 6310

Matrix minima (Least cost entry) method

Various steps of Matrix Minima method are given below:

Step I: Determine the smallest cost in the cost matrix of the transportation table. Let it be C_{ij}. Allocate X_{ij} = Min. (a_i, b_j) in the cell (i, j).

Step II: a) If $X_{ij} = a_i$, cross off the i^{th} row of the transportation table and decrease b_i by a_i and go to Step III.

b) If $X_{ij} = b_j$, cross off the j^{th} column of the transportation table and decrease a_i by b_j and go to Step III.

c) If $X_{ij} = a_i = b_j$, cross of either the i^{th} row or the j^{th} column but not the both.

Step III: Repeat steps I & II for the resulting reduced transportation table until all the requirements are satisfied. Whenever the minimum cost is not unique, make an arbitrary choice among the minima.

Illustration of matrix minima (Least cost entry) method

Let us illustrate this method by considering example 1given earlier in this lesson . The first allocation is made in the cell (3, 4) where the cost of transportation is minimum, the magnitude being X_{34}=50. This satisfies the requirement of S chilling centre. Therefore cross off the fourth column. The second allocation is made in the cell (3,2) having minimum cost among remaining cells and its magnitude being X_{32}=Min.(120,90-50)=40. This satisfies the supply of route C. Cross off the third row. Among the remaining cells, the minimum cost is found in cell (2, 3) so the third allocation is done in cell (2, 3) and its magnitude being X_{23}=Min.(90,160)=90. The requirement of the chilling center R is fulfilled, hence third column is crossed off. Out of the remaining cells the minimum cost is found in cell (1,1) and its magnitude is X_{11}=Min(140,150)=140, as the requirement of chilling center P is exhausted hence column first is deleted. Among remaining two cells minimum cost is found in cell (1,2) and its magnitude X_{12}=Min(80,150-140)=10 and the last allocation is done in the cell (2,2) and its magnitude is X_{22}=Min(80-10,70)=70. Now all requirements have been satisfied and hence an initial basic feasible solution to T.P. has been obtained and given in the following table.

Routes	Chilling centers				Route Capacity
	P	Q	R	S	
A	140 16	10 18	21	12	150
B	17	70 19	90 14	13	160
C	32	40 11	15	50 10	90
Chilling Centre Capacity	140	120	90	50	400

The transportation cost according to the above allocation is given by z = 16 * 140 + 18 * 10 + 19 * 70 + 11 * 40 + 14 * 90 + 10 * 50 = 5950

Illustration of vogel's approximation method (VAM)

Let us illustrate this method by considering Example 1 discussed before in this lesson .

Routes	Chilling centers				Route Capacity	Penalties
	P	Q	R	S		
A	16	18	21	12	150	(4)
B	17	19	14	13	160	(1)
C	32	11	15	10	90	(1)
Chilling Centre Capacity	140	120	90	50	400	
Penalties	(1)	(7) ↑	(1)	(2)		

For each row and column of the transportation table determine the penalties and put them along side of the transportation table by enclosing them in parenthesis against the respective rows and beneath the corresponding columns. Select the row or column with the largest penalty i.e. (7) (marked with an arrow) associated with second column and allocate the maximum possible amount to the cell (3,2) with minimum cost and allocate an amount X_{32}=Min(120,90)=90 to it. This exhausts the capacity of route C. Therefore, cross off third row. The first reduced penalty table will be:

Routes	Chilling centers				Route Capacity	Penalties
	P	Q	R	S		
A	16	18	21	12	150	(4)
B	17	19	14	13	160	(1)
Chilling Centre Capacity	140	30	90	50	310	
Penalties	(1)	(1)	(7) ↑	(1)		

In the first reduced penalty table the maximum penalty of rows and columns occurs in column 3, allocate the maximum possible amount to the cell (2,3) with minimum cost and allocate an amount X_{23}=Min(90,160)=90 to it . This exhausts the capacity of chilling center R .As such cross off third column to get second reduced penalty table as given below.

Routes	Chilling centers			Route Capacity	Penalties
	P	Q	S		
A	16	18	12	150	(4) →
B	17	19	13	70	(4)
Chilling Centre Capacity	140	30	50	220	
Penalties	(1)	(1)	(1)		

In the second reduced penalty table there is a tie in the maximum penalty between first and second row. Choose the first row and allocate the maximum possible amount to the cell (1,4) with minimum cost and allocate an amount X_{14}=Min(50,150)=50 to it . This exhausts the capacity of chilling center S so cross off fourth column to get third reduced penalty table as given below:

Routes	Chilling centers		Route Capacity	Penalties
	P	Q		
A	16	18	100	(2) →
B	17	19	70	(2)
Chilling Centre Capacity	140	30	170	
Penalties	(1)	(1)		

The largest of the penalty in the third reduced penalty table is (2) and is associated with first row and second row. We choose the first row arbitrarily whose min. cost is C_{11} = 16. The fourth allocation of X_{11}=min. (140, 100) =100 is made in cell (1, 1). Cross off the first row. In the fourth reduced penalty table i.e. second row, minimum cost occurs in cell (2,1) followed by cell (2,2) hence allocate X_{21}=40 and X_{22}=30. Hence the whole allocation is as under:

Routes	Chilling centers								Route Capacity
	P		Q		R		S		
A	100	16		18		21	50	12	**150**
B	40	17	30	19	90	14		13	**160**
C		32	90	11		15		10	**90**
Chilling Centre Capacity	140		120		90		50		400

The transportation cost according to the above allocation is given by z = 16 * 100 + 12* 50 + 17 * 40 + 19 * 30 + 14 *90 + 11 * 90 = 5700

Note: Generally, Vogel's Approximation Method is preferred over the other methods because the initial BFS obtained is either optimal or very close to the optimal solution.

Lesson 8
INTRODUCTION AND MATHEMATICAL FORMULATION

8.1 Introduction

In earlier module, transportation problem and the technique of solving such a problem was discussed. In this lesson, the Assignment Problem, which is a special type of transportation problem, is introduced. Here the objective is to minimize the cost or time of completion of a number of jobs by a number of persons. In other words, when the problem involves the allocation of n different facilities to n different tasks, it is often termed as Assignment Problem. The assignment problem deals in allocating the various origins (jobs) to equal number of destinations (persons) on a one to one basis in such a way that the resultant effectiveness is optimized (minimum cost or maximum profit). This is useful in solving problems such as assigning men to different operations in a milk plant, milk tankers to delivery routes, machine operators to machines, jobs to persons in a dairy plant etc.

8.2 Definition of Assignment Problem

Assignment problem is special class of the transportation problem in which the supply in each row represents the availability of a resource such as man, vehicle, product and demand in each column represents different activities to be performed, such as jobs, routes, milk plants respectively is required. The name �Assignment Problem �originates from the classical problem where the objective is to assign a number of origins (jobs) to equal number of destinations(persons) at a minimum cost (or Maximum profit). Suppose there are n jobs to be performed and n persons are available for doing these jobs. Assume that each person can do each job at a time, though with varying degree of efficiency. Let C_{ij} be the cost if i^{th} person is assigned the j^{th} job, the problem is to find an assignment so that the total cost for performing all jobs is minimum. One of the important characteristics of assignment problem is that only one job (or worker) is assigned to one machine (or project). Hence, the number of sources are equal to the number of destinations and each requirement and capacity value is exactly one unit.

The assignment problem can be stated in the form n x n cost matrix [C_{ij}] of real number as given below

Sources (Milk plants)	Jobs					
	J_1	J_2	��	J_j	��	J_n
P_1	C_{11}	C_{12}	��	C_{1j}	��	C_{1n}
P_2	C_{21}	C_{22}	��	C_{2j}	��	C_{2n}
:	:	:	��	:	��	:
P_i	C_{i1}	C_{i2}	��	C_{ij}	��	C_{in}
:	:	:	��	:	��	:
P_n	C_{n1}	C_{n2}	��	C_{nj}	��	C_{nn}

Formulation of an Assignment Problem

Let us consider the case of a milk plant which has three jobs to be done on the three available machines. Each machine is capable of doing any of the three jobs. For each job the cost depends on the machine to which it is assigned. Costs incurred by doing various jobs on different machines are given below

Job	Machine		
	I	II	III
A	7	8	6
B	5	4	9
C	2	5	6

The problem of assigning jobs to machines, one to each, so as to minimize total cost of doing all the jobs, is an assignment problem. Each job machine combination which associates all jobs to machines on one �to-one basis is called an assignment. In the above example let us write all the possible assignments

Number	Assignment	Total Cost
1	Job A-Machine I, Job B �Machine II, Job C-Machine III	7+8+6=21

2	Job A-Machine I, Job B �Machine III, Job C-Machine II	7+9+5=21
3	Job A-Machine II, Job B �Machine III, Job C-Machine I	8+9+2=19
4	Job A-Machine II, Job B �Machine I, Job C-Machine III	8+5+6=19
5	Job A-Machine III, Job B �Machine I, Job C-Machine II	6+5+2=13
6	Job A-Machine III, Job B �Machine II, Job C-Machine I	6+4+2=12

As per the above assignment, the assignment number 6 having total cost 12 is minimum therefore needs to be selected. But selecting assignment in this manner is quite time consuming.

8.3 Mathematical Formulation of Assignment Problem

Using the notations described above, the assignment problem consist of finding the values of X_{ij} in order to minimize the total cost

$$Minimize\ Z = \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}X_{ij} \quad i = 1,2,\ldots,n\,;\, j = 1,2,\ldots,n$$

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Subject to restrictions

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$$X_{ij} = \begin{cases} 1 \text{ if the } i^{th} \text{person is assigned to } j^{th} \text{ job} \\ 0 \text{ if not} \end{cases}$$

$$\sum_{j=1}^{n} X_{ij} = 1(only\ one\ job\ is\ done\ by\ the\ i^{th}\ person)$$

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$$\sum_{i=1}^{n} X_{ij} = 1(only\ one\ person\ should\ be\ assigned\ the\ j^{th}\ job)$$

where X_{ij} denotes the j^{th} job to be assigned to the i^{th} person. An assignment problem could thus be solved by Simplex Method.

We state below, the following theorems which have potential applications in finding out of the optimal solution for assignment problems:

Theorem 1 Reduction Theorem

It states that in an assignment problem, if we add or subtract a constant to every element of any line (row or column) of the cost matrix [C_{ij}], then an assignment that minimizes the total cost on one matrix also minimizes the total cost on the other matrix.

Theorem 2 In an assignment problem with cost (C_{ij}), if all $C_{ij} \geq 0$ then a feasible solution (X_{ij}) which satisfies, $\sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij} X_{ij} = 0$ �is optimal for the problem

Remarks

There are situations when a particular assignment may not be permissible. In such situations assign a very high cost (say M) for such an assignment and proceed as usual.

1. If the assignment problem involves maximization, convert the effective matrix to an opportunities loss matrix by subtracting each element from the highest element of the matrix. Minimization of the resulting matrix is the same as the maximization of the original matrix.

8.4 Similarity of Assignment Problem to Transportation Problem

The assignment problem is a particular case of transportation problem in which a number of operations are to be assigned to an equal number of operators, where each operator performs only one operation. The objective is to maximize overall profit or minimize overall cost for a given assignment schedule.

The Assignment Problem is a special case of the transportation problem in which m=n, All a_i and b_j are unity i.e., The availability and requirement at i^{th} origin and j^{th} destination are unity, and each X_{ij} is limited to one of the two values 0 and 1. Under these circumstances, exactly n of X_{ij} can be non-zero (i.e., unity), one in each row of the table and one in each column.

An assignment problem is a completely degenerate form of a transportation problem. The units available at each origin and units demanded at each destination are all equal to one. This means exactly one occupied cell in each row and each column of the transportation table. i.e., only n occupied cells in place of the required (n + n) = (2n 1).

Hungarian assignment method

The Hungarian method of assignment provides us with an efficient means of finding the optimal solution. The Hungarian method is based upon the following principles:

(i) If a constant is added to every element of a row and/or column of the cost matrix of an assignment problem the resulting assignment problem has the same optimum solution as the original problem or vice versa.

(ii) The solution having zero total cost is considered as optimum solution.

Hungarian method of assignment problem (minimization case) can be summarized in the following steps:

Step I: Subtract the minimum cost of each row of the cost (effectiveness) matrix from all the elements of the respective row so as to get first reduced matrix.

Step II: Similarly subtract the minimum cost of each column of the cost matrix from all the elements of the respective column of the first reduced matrix. This is first modified matrix.

Step III: Starting with row 1 of the first modified matrix, examine the rows one by one until a row containing exactly single zero elements is found. Make any assignment by making that zero in or enclose the zero inside a. Then cross (X) all other zeros in the column in which the assignment was made. This eliminates the possibility of making further assignments in that column.

Step IV: When the set of rows have been completely examined, an identical procedure is applied successively to columns that is examine columns one by one until a column containing exactly single zero element is found. Then make an experimental assignment in that position and cross other zeros in the row in which the assignment has been made.

Step V: Continue these successive operations on rows and columns until all zeros have been either assigned or crossed out and there is exactly one assignment in each row and in each column. In such case optimal assignment for the given problem is obtained.

Step VI: There may be some rows (or columns) without assignment i.e. the total number of marked zeros is less than the order of the matrix. In such case proceed to step VII.

Step VII: Draw the least possible number of horizontal and vertical lines to cover all zeros of the starting table. This can be done as follows:

1. Mark (√) in the rows in which assignments has not been made.
2. Mark column with (√) which have zeros in the marked rows.
3. Mark rows with (√) which contains assignment in the marked column.
4. Repeat 2 and 3 until the chain of marking is completed.
5. Draw straight lines through marked columns.
6. Draw straight lines through unmarked rows.

By this way we draw the minimum number of horizontal and vertical lines necessary to cover all zeros at least once. It should, however, be observed that in all n x n matrices less than n lines will cover the zeros only when there is no solution among them. Conversely, if the minimum number of lines is n, there is a solution.

Step VIII: In this step, we

1. � Select the smallest element, say X, among all the not covered by any of the lines of the table; and
2. Subtract this value X from all of the elements in the matrix not covered by lines and add X to all those elements that lie at the intersection of the horizontal and vertical lines, thus obtaining the second modified cost matrix.

Step IX: Repeat Steps IV, V and VI until we get the number of lines equal to the order of matrix I, till an optimum solution is attained.

Step X: We now have exactly one encircled zero in each row and each column of the cost matrix. The assignment schedule corresponding to these zeros is the optimum assignment. The above technique is explained by taking the following examples

Example 1

A plant manager has four subordinates, and four tasks to be performed. The subordinates differ in efficiency and the tasks differ in their intrinsic difficulty. This estimate of the times each man would take to perform each task is given in the effectiveness matrix below.

	I	II	III	IV

A	8	26	17	11
B	13	28	4	26
C	38	19	18	15
D	19	26	24	10

How should the tasks be allocated, one to a man, so as to minimize the total man hours?

Solution

Step I : Subtracting the smallest element in each row from every element in that row, we get the first reduced matrix.

0	18	9	3
9	24	0	22
23	4	3	0
9	16	14	0

Step II: Next, we subtract the smallest element in each column from every element in that column; we get the second reduced matrix.

Step III: Now we test whether it is possible to make an assignment using only zero distances.

0	14	9	3
9	20	0	22
23	0	3	0
9	12	14	0

(a) Starting with row 1 of the matrix, we examine rows one by one until a row containing exactly single zero elements are found. We make an experimental assignment (indicated by) to that cell. Then we cross all other zeros in the column in which the assignment was made.

(b) When the set of rows has been completely examined an identical procedure is applied successively to columns. Starting with Column 1, we examine columns until a column containing exactly one remaining zero is found. We make an experimental assignment in that position and cross other zeros in the row in which the assignment was made. It is found that no additional assignments are possible. Thus, we have the complete Zero assignment,

A I, B III, C II, D IV

The minimum total man hours are computed as

Optimal assignment	Man hours

A I	8
B III	4
C II	19
D IV	10
Total	**41 hours**

Example 2

A dairy plant has five milk tankers I, II, III, IV & V. These milk tankers are to be used on five delivery routes A, B, C, D, and E. The distances (in kms) between dairy plant and the delivery routes are given in the following distance matrix

	I	II	III	IV	V
A	160	130	175	190	200
B	135	120	130	160	175
C	140	110	155	170	185
D	50	50	80	80	110
E	55	35	70	80	105

How the milk tankers should be assigned to the chilling centers so as to minimize the distance travelled?

Solution

Step I: Subtracting minimum element in each row we get the first reduced matrix as

30	0	45	60	70
15	0	10	40	55
30	0	45	60	75
0	0	30	30	60
20	0	35	45	70

Step II: Subtracting minimum element in each column we get the second reduced matrix as

30	0	35	30	15
15	0	0	10	0
30	0	35	30	20
0	0	20	0	5
20	0	25	15	15

Step III: Row 1 has a single zero in column 2. We make an assignment by putting around it and delete other zeros in column 2 by marking X. Now column1 has a single zero in column 4 we make an assignment by putting and cross the other zero which is not yet crossed. Column 3 has a single zero in row 2; we make an assignment and delete the other zero which is uncrossed. Now we see that there are no remaining zeros; and row 3, row 5 and column 4 has no assignment. Therefore, we cannot get our desired solution at this stage.

30	$\boxed{0}$	35	30	15		√
15	$\cancel{0}$	$\boxed{0}$	10	$\cancel{0}$	L_2	
30	$\cancel{0}$	35	30	20		√
$\boxed{0}$	$\cancel{0}$	20	$\cancel{0}$	5	L_3	
20	$\cancel{0}$	25	15	15		√
	√ L_1					

Step IV: Draw the minimum number of horizontal and vertical lines necessary to cover all zeros at least once by using the following procedure

1. Mark (√) row 3 and row 5 as having no assignments and column 2 as having zeros in rows 3 and 5.
2. Next we mark (√) row 2 because this row contains assignment in marked column 2. No further rows or columns will be required to mark during this procedure.
3. Draw line L_1 through marked col.2.
4. Draw lines L_2 & L_3 through unmarked rows.

Step V: Select the smallest element say X among all uncovered elements which is X = 15. Subtract this value X=15 from all of the values in the matrix not covered by lines and add X to all those values that lie at the intersections of the lines L_1, L_2 & L_3.
Applying these two rules, we get a new matrix

15	0	20	15	0
15	15	0	10	0
15	0	20	15	5
0	15	20	0	5
5	0	10	0	0

Step VI: Now reapply the test of Step III to obtain the desired solution.

15	$\cancel{0}$	20	15	$\boxed{0}$
15	15	$\boxed{0}$	10	$\cancel{0}$
15	$\boxed{0}$	20	15	5
$\boxed{0}$	15	20	$\cancel{0}$	5
5	$\cancel{0}$	10	$\boxed{0}$	$\cancel{0}$

The assignments are

A → V   B → III   C → II   D → I   E → I

Total Distance 200 + 130 + 110 + 50 + 80 = 570

INTRODUCTION AND ELEMENTARY CONCEPTS

12.1 Introduction

The replacement problems are concerned with the situations that arise when some items such as machines, men, electric appliance etc. need replacement due to their decreased efficiency, failure or breakdown. The deteriorating efficiency or complete breakdown may be either gradual or all of a sudden. A replacement is called for whenever new equipment offers more efficient or economical service than the existing one. The problem in such situation is to determine the best policy to be adopted with respect to replacement of the equipment. In case of items whose efficiency go on decreasing according to their age, we have to spend more and more money on account of increased operating cost, increased repair cost, increased scrap, etc. In such cases the replacement of an old item with a new one is the only alternative to prevent such increased expenses. Thus, it becomes necessary to determine the age at which replacement is more economical rather than continuing with the same.

12.2 Types of Replacement Situations

The replacement situations may be classified into four categories:

a) Replacement of items that become worse with time e.g. milk plant machinery, tools, vehicles, equipment etc.
b) Replacement of items which do not deteriorate with time but break down completely after certain usage e.g. electric tubes, machinery parts etc.
c) Replacement of items that becomes obsolete due to new developments.
d) The existing working staff in an organization gradually reduces due to death, retirement and other reasons.

The problem is to decide the best policy to adopt with regard to replacement. The need for replacement arises in a number of different situations so that different types of decisions may have to be taken. For example:

a) It may be necessary to decide whether to wait for certain items to fail, which might cause some loss, or to replace the same in advance, even at a higher cost.
b) An item can be considered individually to decide whether or not to replace immediately.
c) It is necessary to decide whether to replace by the same item or by an improved type of item.

12.3 Types of Failure

There are two types of failure: i) Gradual failure ii) Sudden failure

12.3.1 Gradual failure

It means slow or progressive failure as the life of the item increases, its efficiency decreases resulting in decreased productivity, increased operating cost and decrease in the value of the item, e.g. machines/equipment etc.

12.3.2 Sudden failure

In this type of failure the items do not deteriorate markedly with service but which ultimately fail after some period of usage, thus precipitating cost of failure. Sometimes sudden failure of an item may cause loss of production or may also account for damaged or faulty products. The period between installation and failure is not constant for any particular type of equipment but will follow some probability distribution which may be progressive, retrogressive or random in nature.

12.3.2.1 Progressive failure

Under this mechanism, the probability of failure increases with the increase in the life of an item.

12.3.2.2 Retrogressive failure

Certain items have more probability of failure in the beginning of their life and as time passes, the chances of failure become less. In other words, the ability of the unit to survive the initial period of life increases its expected life.

12.3.2.3 Random failure

Under this mechanism, constant probability of failure is associated with items that fail from random causes such as physical shocks, not related to age.

12.4 Assumptions

Following assumptions are essentially required for replacement decisions:

i) The quality of the output remains constant.
ii) Replacement and maintenance costs remain constant.
iii) The operational efficiency of the equipment remains constant.

iv) There is no change in technology of the asset under consideration.

12.5 OR Methodology of Solving Replacement Problem

OR provides a methodology for tackling replacement problem which is discussed below:

i) Identify the items to be replaced and also their failure mechanism.

ii) Collect the data relating to the depreciation cost and the maintenance cost for the items which follow gradual failure mechanism. In case of sudden failure of items, collect the data for replacement cost of the failed items.

iii) Select a suitable replacement model as discussed in Lesson 13.